Mathematics for Engineers II. lectures

Pál Burai

Fourier transform

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Fourier transform, heuristic

Let's consider the Fourier series of a periodic function f with period $2L$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

Taking the limit $L \to \infty$ the harmonics $\omega_n = \frac{n\pi}{L}$ are getting thicker, that is, $\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L} \to 0$ if $n \to \infty$.

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$\cos(x)\cos(y) + \sin(x)\sin(y) = cos(x-y)$ we get

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\pi}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}(t-x)\right)dt.$$

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The first term tends to zero as $L \to \infty$, the second term is a Riemann integral sum of the function $\omega \mapsto \frac{1}{\pi}\int_{-L}^{L} f(t)\cos(\omega(t-x))\,dt$ with nodes ω_n and with the step size $\Delta\omega$.

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Inserting the integral instead of the Riemann sum we have

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Applying the previous trigonometric identity we get:

$$f(x) = \int_0^\infty (a_\omega \cos(\omega x) + b_\omega \sin(\omega x)) d\omega, \qquad \textbf{(Fourier integrál)}$$

where

$$a_\omega = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(t) \cos(\omega t) dt, \qquad b_\omega = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(t) \sin(\omega t) dt.$$

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As earlier, if f is even, then $b_\omega = 0$, and if f is odd, then $a_\omega = 0$.

Fourier transform

Since $\cos(\omega(t-x))$ is even we can integrate over the interval$[0,\infty]$ instead of $[-\infty,\infty]$, dividing the result by two we get

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Subtracting i times the latter from the former, using Euler's formula, we get the **complex form of the Fourier integral**:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left(\int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt\right)d\omega.$$

Definition

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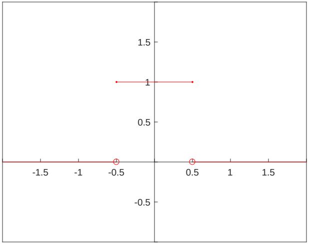
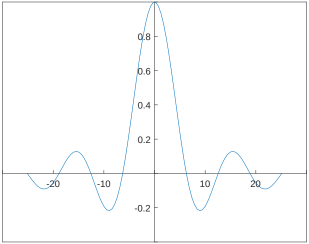
$$\mathcal{F}^{-1}[F](t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

is called teh **inverse Fourier transform of** F.

Let

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{then} \qquad \mathcal{F}[f](\omega) = \frac{2}{\omega} \sin \frac{\omega}{2}.$$



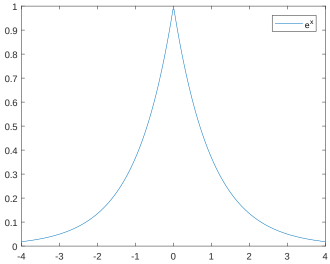


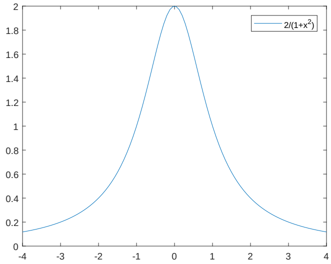
Fourier transform, Examples

Let $f_\gamma(x) = e^{-\gamma|x|}$, where $\gamma > 0$ is given. Then

$$\mathcal{F}[f_\gamma](\omega) = F_\gamma(\omega) = \frac{2\gamma}{\gamma^2 + \omega^2}.$$

The graph of of f_1 and F_1.

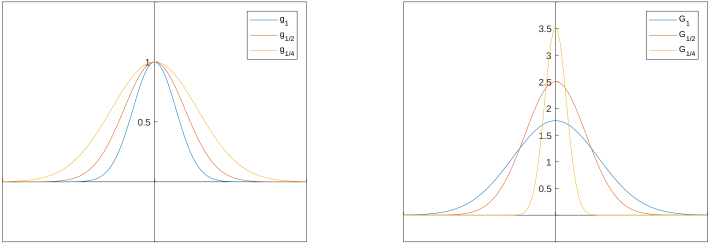




Fourier transform, Examples

Determine the Fourier transform of the Gauss function $g_a(x) = e^{-ax^2}$, where a is a real parameter.

$$\mathcal{F}[g_a](\omega) = G_a(\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$



Theorem, Properties of Fourier transform

Let f and g be absolutely integrable functions. Denote by F and G their Fourier transform respectively. Then

- $$\mathcal{F}[af(x) + bg(x)](\omega) = aF(\omega) + bG(\omega)$$

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  for arbitrarily fixed x_0;
- $$\mathcal{F}[x^n f(x)](\omega) = i^n F^{(n)}(\omega), \quad \text{és} \quad \mathcal{F}[f^{(n)}(x)](\omega) = (i\omega)^n F(\omega)$$
  for an arbitrary natural number n.

Properties of Fourier transform

Definition

Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be given functions, then the function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

is called the **convolution** of f and g. (We assume that the integral above exists.)

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Theorem

Assuming the existence of the corresponding integrals, the Fourier transform of the convolution is the product of the Fourier transforms, that is to say,

$$\mathcal{F}[(f * g)(x)](\omega) = \mathcal{F}[f(x)](\omega) \cdot \mathcal{F}[g(x)](\omega).$$

1. Let's define the following transformations, translation, modulation and dilatation:

$$(\tau_h f)(x) = f(x+h), \qquad (\nu_\Omega f)(x) = e^{i\Omega x} f(x), \qquad (\delta_a f)(x) = f(ax).$$

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   Using only $\mathcal{F}[f]$, express the previous transformations!
2. Using only $\mathcal{F}^{-1}[F]$ express the the previous transformations!
3. Using only $\mathcal{F}[f] = F$ obtain the Fourier transform of the following functions!

   $$f(2t-3), \qquad f(2(x-3)), \qquad (x^2 f(3x))'', \qquad x^3 f''(x-3).$$

Discrete Fourier transform

Definition

Let $T > 0$ and $N \in \mathbb{N}$ be given, and
$f_n = f(t_n),\ t_n := nT,\ n = 0, \ldots, N-1$ be a given vector, then the vector

$$F_k = F(\omega_k) = \sum_{n=0}^{N-1} f_n e^{-i\omega_k t_n}, \qquad \omega_k = \frac{2\pi k}{NT}, \qquad k = 0, 1, \ldots, N-1$$

is said to be the **discrete Fourier transform of the vector** $(f_0, \ldots, f_{N-1})$. In notation: DFT.

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Remark

The Discrete Fourier Transform (DFT) is the equivalent of the continuous Fourier Transform for signals known only at N instants separated by sample times T.

For the sake of the numerical calculations it is handier to write the previous equations in matrix form:

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & W^3 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \dots & \vdots & & \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \dots & W^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix},$$

where $W = e^{\frac{-i2\pi}{N}}$.

Discrete Fourier transform

Example

Let the continuous signal be

$$f(t) = 5 + 2\cos(2\pi t - 90^\circ) + 3\cos(4\pi t).$$

Let us sample $f(t)$ at 4 times per second from $t = 0$ to $t = \frac{3}{4}$.

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so

$$f_0 = 8, \quad f_1 = 4, \quad f_2 = 8, \quad f_3 = 0.$$

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Then $W = e^{\frac{-i2\pi}{4}} = -i$, therefore

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ -4i \\ 12 \\ 4i \end{bmatrix}.$$

Inverse discrete Fourier transform

Definition

The **inverse Fourier transform** of

$$F_k = F(\omega_k) = \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi}{N}nk}$$

is

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{\left(\frac{i2\pi}{N}nk\right)},$$

i.e. the inverse matrix is $\frac{1}{N}$ times the complex conjugate of the original (symmetric) matrix.

Fast Fourier transform

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Highly efficient computer algorithms for estimating Discrete Fourier Transforms have been developed since the mid-60's. These are known as Fast Fourier Transform (FFT) algorithms and they rely on the fact that the standard DFT involves a lot of redundant calculations.

Fast Fourier transform

Let $W_N = e^{\frac{-i2\pi}{N}}$, then

$$F_k = \sum_{n=0}^{N-1} f_n e^{\frac{-i2\pi}{N}nk} = \sum_{n=0}^{N-1} f_n W_N^{nk}.$$

It is easy to realise that the same values of W_N^{nk} are calculated many times as the computation proceeds. Firstly, the integer product nk repeats for different combinations of n and k; secondly, $W_N^{nk}$is a periodic function with only N distinct values.

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It is easy to realise that the same values of W_N^{nk} are calculated many times as the computation proceeds. Firstly, the integer product nk repeats for different combinations of n and k; secondly, $W_N^{nk}$is a periodic function with only N distinct values. From now on N assumed to be even. Splitting the single summation over N samples into 2 summations, each with $\frac{1}{N}$ samples, one for n even and the other for n odd. Substitute $m = \frac{n}{2}$ for n even and $m = \frac{n-1}{2}$ for n odd and write:

$$F_k = \sum_{m=0}^{\frac{N}{2}-1} f_{2m} W_N^{2mk} + \sum_{m=0}^{\frac{N}{2}-1} f_{2m+1} W_N^{(2m+1)k} = \sum_{m=0}^{\frac{N}{2}-1} f_{2m} W_{\frac{N}{2}}^{mk} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_{2m+1} W_{\frac{N}{2}}^{mk},$$

since $W_N^{2mk} = e^{-i\frac{2\pi}{N}2mk} = e^{-i\frac{2\pi}{\frac{N}{2}}mk} = W_{\frac{N}{2}}^{mk}$. So, the transformation of a vector with length N can be expressed by transformed vectors with length $\frac{N}{2}$.

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$$F_k = G_k + W_N^k H_k.$$

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Thus the N-point DFT can be obtained from two $\frac{N}{2}$-point transforms, one on even input data, G_k, and one on odd input data, H_k. Although the frequency index k ranges over N values, only $\frac{N}{2}$ values of G_k and H_k need to be computed since they are periodic in k with period $\frac{N}{2}$. Hence it is reasonable to take a sample with length 2^j (N is an integral power of 2).

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N	N^2 (DFT)	$\frac{N}{2}\log_2 N$ (FFT)	Saving
32	1024	80	92%
256	65536	1024	98%
1024	1048576	5120	99.5%

Fast Fourier transform, Example

Consider $N = 8$

$$W_8^0 = 1, \qquad W_8^1 = \frac{1-i}{\sqrt{2}} =: a, \qquad W_8^2 = a^2 = -i, \qquad W_8^3 = a^3 = -ia,$$

$$W_8^4 = a^4 = -1, \qquad W_8^5 = a^5 = -a, \qquad W_8^6 = a^6 = i, \qquad W_8^7 = a^7 = ia.$$

Így

$$\begin{aligned}
F_0 &= G_0 + W_8^0 H_0 \\
F_1 &= G_1 + W_8^1 H_1 \\
F_2 &= G_2 + W_8^2 H_2 \\
F_3 &= G_3 + W_8^3 H_3 \\
F_4 0 &= G_0 + W_8^4 H_0 &&= G_0 - W_8^0 H_0 \\
F_5 &= G_1 + W_8^5 H_1 &&= G_1 - W_8^1 H_1 \\
F_6 &= G_2 + W_8^6 H_2 &&= G_2 - W_8^2 H_2 \\
F_7 &= G_3 + W_8^7 H_3 &&= G_3 - W_8^3 H_3
\end{aligned}$$